TITLE: Bispectra and Phase Correlations for Chaotic Dynamical Systems

BISPECTRA AND PHASE CORRELATIONS FOR CHAOTIC DYNAMICAL SYSTEMS

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The bispectrum is the natural third-order generalisation of the power spectrum. It provides information about correlations between different Fourier components of a signal or image, and about the statistics of Fourier phase. A number of numerical and experimental studies of the bispectra of chaotic systems have been published. In this paper we present the first analytical calculations of the bispectra of chaotic dynamical systems. First, for a generalisation of the classical sawtooth or Renyi map, we calculate the bispectrum using symbolic dynamics. Also, for intermittent systems, we calculate the bispectrum using the relationship between these systems and renewal processes. We review the results of these calculations, drawing some conclusions about the characteristic features of the bispectra of chaotic systems, and compare them with the features of some financial time series.

Key words: Chaos, bispectrum, time series, intermittent, finance.

1 Introduction: power spectrum and phase in time-series analysis

Given an experimental time series x_t, what can we say about the nature of the system that produced it? This is a question commonly encountered by experimental scientists, and also by those who deal with financial time series such as stock prices or currency exchange rates. Often the first attack at the question is to calculate the power spectrum $P(\omega)$. The power spectrum reveals how the variance, or power, in the time series is shared between the different frequencies ω.

Because the power spectrum contains no phase information, it is not possible to reconstruct the original signal x_t from it, and there are some questions that can not be answered by inspecting only the power spectrum. One of these is the question of whether an apparently random signal originates from chaos in a low-dimensional deterministic dynamical system or from a stochastic process.[1,2]

The Fourier transform

$$\tilde{x}(\omega) = \sum_t x_t e^{i\omega t} \tag{1}$$

contains the same information as the original signal, and $\tilde{x}$ can be written in terms of a modulus and a phase:

$$\tilde{x}(\omega) = |\tilde{x}(\omega)|\, e^{i\phi(\omega)}, \tag{2}$$

where the modulus $|\tilde{x}(\omega)|$ determines the power spectrum. To go beyond the power spectrum while still using the concepts of Fourier analysis, one should therefore consider the statistics of the Fourier phase $\phi(\omega)$. In fact, the phase is more important in determining the appearance of an image than the Fourier modulus or power spectrum.[3] All information about the location of features in an image or the time of occurrence of features in a signal is contained in the phase.

It is not straightforward to study the statistics of phase in a time series. One advantage of the power spectrum is that it is a well-defined statistical average, being the Fourier transform of the autocorrelation function. However, stationary stochastic processes (including those defined by a chaotic dynamical system with its invariant measure) do not possess a well-defined average phase $\phi(\omega)$ or $e^{i\phi(\omega)}$ for any frequency ω.

We can understand why the Fourier phase factor $e^{i\phi(\omega)}$ does not have an average value for a stationary process by noting that translating the original time series by a delay s (that is, replacing x_t by $x_{(t-s)}$) is equivalent to multiplying each Fourier phase factor by $e^{i\omega s}$. This is just a restatement of the 'shift theorem' of Fourier analysis. Since any time-average of a stationary process must be invariant under a time-shift, this explains why the phase factor does not have a well-defined average.

This argument also provides a hint of how we might construct well-defined average quantities that contain phase information. For any two frequencies ω_1 and ω_2, the phase product

$$R(\omega_1, \omega_2) = e^{i\phi(\omega_1)} e^{i\phi(\omega_2)} e^{-i\phi(\omega_1+\omega_2)} \tag{3}$$

is invariant under a translation. A non-zero average of this quantity implies that the phases of the Fourier components at frequencies ω_1, ω_2 and $\omega_1 + \omega_2$ are correlated.

In fact, the mean value of R is the phase of the bispectrum,[4,5] evaluated at the frequencies (ω_1, ω_2). Just as the power spectrum $P(\omega)$ is the Fourier transform of the autocorrelation function, the bispectrum $P(\omega_1, \omega_2)$ is the double Fourier transform of the bi-correlation function

$$C_x(s,t) = \langle (x_s - \bar{x})(x_t - \bar{x})(x_0 - \bar{x}) \rangle \,, \tag{4}$$

where the angled brackets $\langle\rangle$ denote an expectation value.

The bispectrum can also be estimated directly from the Fourier transform:

$$P(\omega_1, \omega_2) = \lim_{T\to\infty} \frac{1}{2T} \langle \tilde{x}_T(\omega_1) \tilde{x}_T(\omega_2) \tilde{x}_T^*(\omega_1 + \omega_2) \rangle \,, \tag{5}$$

where the subscript T implies that the sum in equation (1) is over the first T samples in the time series, and $*$ denotes complex conjugation.

Higher-order generalisations of the power spectrum, known as polyspectra, can also be defined[4,5]. The bispectrum has been applied in time-series analysis to construct tests[6,7,8]for nonlinearity and for normality (meaning the presence of a normal distribution). These tests depend upon the observations that for normal systems, the bispectrum is zero, and for linear systems (that is, processes constructed by a linear operator acting upon an i.i.d. process) the bicoherence

$$b(\omega_1, \omega_2) = \frac{|P(\omega_1, \omega_2)|^2}{P(\omega_1)P(\omega_2)P(\omega_1 + \omega_2)} \tag{6}$$

is constant.

The bispectrum has also been used to study phase correlations and the interactions between Fourier components in many experimental systems, including electroencephalographic signals from the human brain,[9] and the oscillations of the earth.[10] A number of authors[8,11,12,13] have studied the bispectra of time series from chaotic dynamical systems, either by numerical simulation or experimental

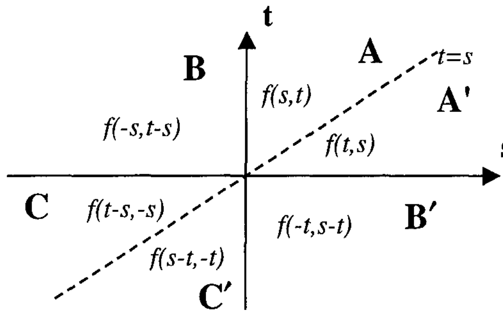

Figure 1. Symmetries of the bicorrelation function. For example, if the bicorrelation is given by $f(s,t)$ in the region A, then the bicorrelation in region C is given by $f(t-s,s)$.

measurement. Subba Rao[8] identified 'ridges' in the bicoherence as a characteristic of chaotic systems, and other authors studying electrical circuits have noted various changes in the bispectral characteristics of time series as the systems generating them make the transition to chaos.

To date, there are no published examples of chaotic dynamical systems where the bispectrum can be calculated exactly. In this paper we present an example of such a calculation, for a simple discrete-time system which is a generalisation of the well-known saw-tooth or Renyi map. We also show how the relationship between intermittent chaotic systems and renewal processes can be used to give asymptotic low-frequency expressions for the bispectra of intermittent systems. We identify some common features of these two families which may be bispectral characteristics of chaos. Finally, we compare some financial time series with the chaotic examples. Before presenting the calculations, we briefly review some properties of the bicorrelation function and bispectrum.

2 Symmetries of the bicorrelation function and calculation of the bispectrum

This section summarises some properties of the bicorrelation function and bispectrum that are useful in deriving the form of the bispectrum. From its definition in equation (4), one can see that, for a stationary process, the bicorrelation function $C_x(s,t)$ has the symmetries

$$C_x(s,t) = C_x(t,s) \qquad C_x(-s,t) = C_x(s,s+t). \tag{7}$$

By composing these symmetry operations, one can reach any part of the (s,t) plane from the region $0 < s < t$ where the bicorrelation is most easily calculated. These relationships are summarised in figure 1.

The bispectrum $P(\omega_1,\omega_2)$ is defined as the double Fourier transform of the bicorrelation function, the Fourier integral being evaluated over the entire (s,t) plane. Let $P_A(\omega_1,\omega_2)$ be the integral over only region A in figure 1. If the value of

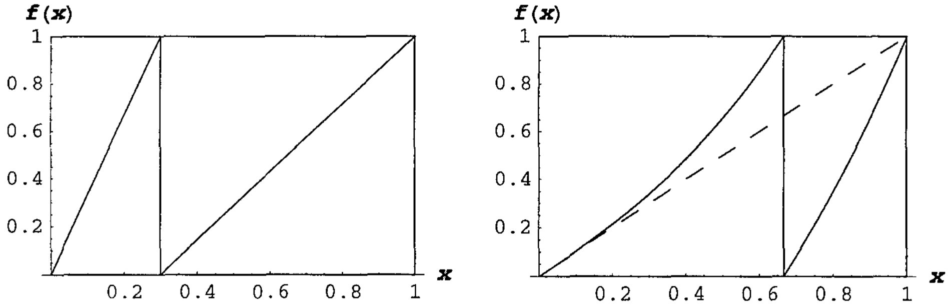

Figure 2. The two types of dynamical system considered in this paper: the generalised sawtooth map (left) with $b = 0.3$, and the intermittent dynamical system described by equation (28) with $\mu = 1$, $z = 2.7$.

the bicorrelation in region A is given by $C_x(s,t) = \beta(s,t)$, then

$$P_A(\omega_1, \omega_2) = \int_0^\infty ds \quad \int_s^\infty dt \quad \beta(s,t)e^{i(s\omega_1 + t\omega_2)}. \tag{8}$$

Similarly, the integral over region B is

$$P_B(\omega_1, \omega_2) = \int_{-\infty}^0 ds \quad \int_0^\infty dt \quad C_x(s,t)e^{i(s\omega_1 + t\omega_2)}. \tag{9}$$

Using the symmetries in equation (7), and changing the variables of integration, one can show that

$$P_B(\omega_1, \omega_2) = P_A(-(\omega_1 + \omega_2), \omega_2). \tag{10}$$

Similar relationships can be derived between P_A and the Fourier integrals over the other regions in figure 1. The bispectrum can then be calculated by summing all the integrals. The result is

$$P(\omega_1, \omega_2) = \hat{P}_A(\omega_1, \omega_2) + \hat{P}_A(-(\omega_1 + \omega_2), \omega_2) + \hat{P}_A(\omega_1, -(\omega_1 + \omega_2)), \tag{11}$$

where the hat operation $\hat{}$ denotes symmetrisation:

$$\hat{f}(x,y) = f(x,y) + f(y,x). \tag{12}$$

3 The generalised sawtooth map

3.1 The dynamical system

Our first dynamical system is defined by the mapping of the interval $(0,1)$ onto itself

$$f(x) = \begin{cases} \frac{x}{b} & x \leq b \\ \frac{x-b}{1-b} & x > b \end{cases} \tag{13}$$

as pictured in Figure 2. The dynamical system $x_{t+1} = f(x_t)$ has a uniform invariant distribution. It is chaotic, with Lyapounov exponent

$$\lambda = -b\ln b - (1-b)\ln(1-b) \tag{14}$$

easily calculated by standard methods.[14]

This dynamical system is a generalisation of the sawtooth or Renyi map which has been studied by many authors.[14] When $b = 1/2$, the two systems are identical. This generalisation makes the study of the system more difficult, but it is necessary in this case because when $b = 1/2$, the system's bispectrum is zero. This is an effect of the symmetry of the function f. Sakai and Tokumaru[15] have calculated the autocorrelation function of the $b = 1/2$ system using a relationship between this system and an autoregressive process.

3.2 Symbolic dynamics

The method we use to study this system is symbolic dynamics [14], where a sequence of symbols from a discrete and finite alphabet represents a trajectory of a dynamical system. The representation for the generalised sawtooth map is not difficult to derive. Suppose that x_0 and x_1 are two successive points in the trajectory of the system, so that $x_1 = f(x_0)$. Then we have either

$$x_0 = bx_1 \tag{15}$$

or, if $x_0 > b$,

$$x_0 = b + (1-b)x_1. \tag{16}$$

Putting $z_1 = 0$ in the first case, and $z_1 = 1$ in the second, we can write

$$x_0 = z_1 b + x_1 e_1, \tag{17}$$

where $e_i = (b - 2z_i b + z_i)$. Continuing along the same lines, we obtain

$$\begin{aligned} x_0 &= z_1 b + e_1(z_2 b + e_2(z_3 b + e_3 x_3)) \\ &= be_1e_2\cdots e_{n-1}x_n + b\sum_{k=1}^{n} e_1e_2\cdots e_{k-1}z_k. \end{aligned} \tag{18}$$

It is easy to show that, under the (uniform) invariant measure, the z_i are independent random variables, taking the values 0 or 1 with probabilities b and $(1-b)$ respectively. We can therefore use this representation to calculate various averages. For example, in the next section we calculate the autocorrelation function and the power spectrum.

3.3 The power spectrum

Multiplying equation (18) by x_n, we obtain

$$x_0x_n = be_1e_2\cdots e_{n-1}x_n^2 + b\sum_{k=1}^{n} e_1e_2\cdots e_{k-1}z_kx_n. \tag{19}$$

We can now take the expectation value, using the fact that random variables with different subscripts are independent, and the expectation values $\langle z_i \rangle = (1-b)$, $\langle z_i \rangle = (1-b)$, $\langle x_i \rangle = 1/2$, $\langle x_i^2 \rangle = 1/3$ and

$$\langle e_i \rangle = b^2(1-b)^2 = B, \tag{20}$$
$$\langle e_i^2 \rangle = b^3 + (1-b)^3 = C, \tag{21}$$
$$\tag{22}$$

After summing a geometric series, we obtain

$$\langle x_0 x_n \rangle = \frac{1}{4} + \frac{B^n}{12}. \tag{23}$$

This equation is valid for $n \geq 0$. The autocorrelation function is therefore

$$C_x(t) = \langle x_0 x_t \rangle - \langle x_0^2 \rangle = \frac{B^{|t|}}{12}. \tag{24}$$

The power spectrum is the Fourier transform of this function (using the convention of equation (1)):

$$P(\omega) = \frac{1+B^2}{12(1+B^2-2B\cos\theta)}. \tag{25}$$

3.4 The bispectrum

The same methods can be used to calculate the three-point average $\langle x_0 x_s x_t \rangle$ when $0 \leq s \leq t$. We can then calculate the bicorrelation function. The details of this calculation occupy far too much space to be recorded here. The result is, again for $0 \leq s \leq t$,

$$C_x(s,t) = \frac{(1-2b)}{12}\left(B^t - B^s C^{t-s}\right), \tag{26}$$

where B and C are defined in equations (20) and (21). The bispectrum can now be calculated using the method described in the last section. The result is given by equation (11), with

$$P_A(x,y) = \frac{(2b-1)(b-1)be^{-ix}}{12\left(1-Be^{-ix}\right)\left(1-Ce^{-ix}\right)\left(1-Be^{iy}\right)} \tag{27}$$

Contour diagrams of the bispectrum and bicorrelation function for a typical value of b are shown in figure 3. The 'ridges' found by Subba Rao[8] are not evident for this system for any value of b that we have examined. The most noticeable consistent feature is that the energy in the bispectrum is spread over a broad ranges of frequencies rather than being concentrated into narrow bands. This may be a characteristic of fully-developed chaos.

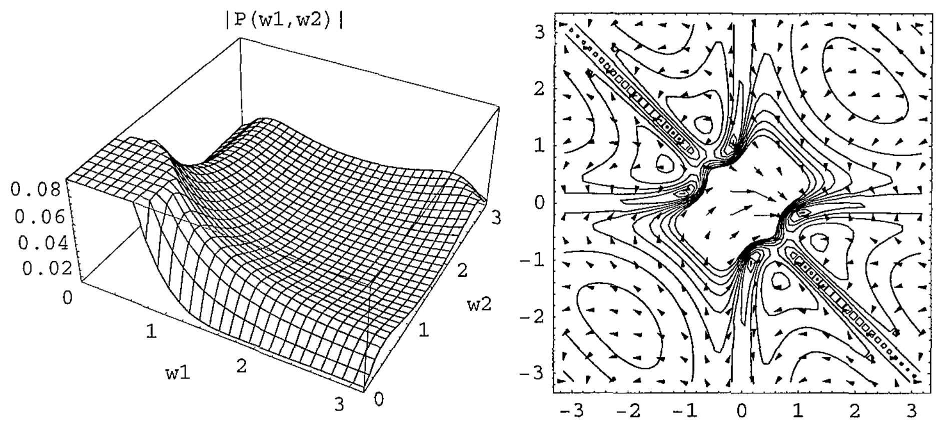

Figure 3. The bispectrum for the sawtooth map with $b = 0.27$. The left graph shows a surface map of the modulus of the bispectrum, evaluated over the region where ω_1 and ω_2 are positive. The right graph shows a direction field displaying the complex phase of the bispectrum overlaid with a contour map, evaluated over both positive and negative frequencies.

4 Intermittent dynamical systems

In this section, we study the bispectra of a class of intermittent dynamical systems. For definiteness, we take as our model the system defined by the mapping

$$f(x) = (x + \mu x^z) \bmod 1, \tag{28}$$

shown in figure 2 for $\mu = 1$, $z = 1.7$. The analysis given here depends only on the form of $f(x)$ close to the intermittent point at $x = 0$, and so applies to a broad class of intermittent dynamical systems,[14] as described by Ben-Mizrachi *et al.*[16] The defining characteristic of these intermittent systems is that they show brief periods of random behaviour between long periods when the system is 'stagnant', remaining close to the centre of intermittency (in the case of the model system above, close to the point $x = 0$).

We will be studying the long-time behaviour, or equivalently the low-frequency behaviour, of these dynamical systems. For this purpose, following Ben-Mizrachi *et al*,[16] we replace the dynamical system's time series x_t by a series of delta-functions located at the times of escape from periods of stagnation. The dynamical system is thus replaced by a renewal process with waiting-time distribution defined by the distribution of waiting times between escape events. As Ben-Mizrachi *et al*[16] show, this distribution can be approximated for large τ by

$$p_\tau(\tau) = \begin{cases} (\alpha - 1)t^{-\alpha} & t \geq 1 \\ 0 & t < 1, \end{cases} \tag{29}$$

where $\alpha = z/(z-1)$. Ben-Mizrachi *et al*[16] also show that the function $c(t)$ giving the probability of an escape event at time t given the occurrence of such an event at time 0 is given by

$$\bar{c}(s) = \frac{1}{1 - \bar{p}_\tau(s)}, \tag{30}$$

where the bar $\bar{\ }$ denotes the Laplace transform. The Laplace transform of p_τ is an incomplete Gamma function,[17] whose small-s behaviour is given by

$$1 - \bar{p}_\tau(s) \sim s^{\alpha-1}\Gamma(2-\alpha) \tag{31}$$

for $1 < \alpha < 2$. For the sake of brevity, we consider only this case (corresponding to $z > 2$) in this paper. Other cases will be covered in a longer paper currently in preparation.

For the case $1 < \alpha < 2$, for small s we have $\bar{c}(s) \sim \Gamma(2-\alpha)s^{1-\alpha}$, and the low-frequency behaviour of the power spectrum[16] is given by

$$P(\omega) = |\bar{c}(i\omega) + \bar{c}(-i\omega)|^2 \sim \frac{\text{constant}}{\omega^{2\alpha-2}}. \tag{32}$$

The bispectrum can also be calculated from $\bar{c}(s)$. First, note that for large t_1 and $(t_2 - t_1)$ and for $0 < t_1 < t_2$,

$$\langle x_{t_1} x_0 \rangle \sim c(\infty)c(t_1). \tag{33}$$

and

$$\langle x_{t_1} x_{t_2} x_0 \rangle \sim c(\infty)c(t_1)c(t_2 - t_1). \tag{34}$$

The bicorrelation $C_x(t_1, t_2)$ is therefore given by

$$C_x(t_1,t_2) = c(\infty)c(t_1)c(t_2-t_1) - c(\infty)^2\left(c(t_1) + c(t_2) + c(t_2 - t_1)\right) + 2c(\infty)^3. \tag{35}$$

After applying the double Laplace transform and taking the limit $s \to 0$, we find that the first term dominates and

$$\bar{C}_x(s_1, s_2) \sim \frac{c(\infty)}{\Gamma^2(2-\alpha)s_1^{\alpha-1}\left(s_2 - s_1\right))^{\alpha-1}}. \tag{36}$$

By setting $s_1 = i\omega_1$, $s_2 = i\omega_2$, we obtain an expression for the bispectrum integral P_A:

$$P_A(\omega_1, \omega_2) \sim \frac{c(\infty)}{\Gamma^2(2-\alpha)\left(i\omega_1\left(\omega_2 - \omega_1\right)\right)^{\alpha-1}}. \tag{37}$$

The bispectrum is then given by equation (11).

Bispectra calculated numerically from time series generated directly by the dynamical system (28) are in good agreement with this result, although a large number of data points is required for the average to converge to a smooth function. Figure 4 shows the result of one comparison.

The structure of these low-frequency bispectra is simple. They show power-law singularities at the lines $\omega_1 = 0$, $\omega_2 = 0$ and $\omega_1 + \omega_2 = 0$. Like the bispectra for the sawtooth map, the bispectra for intermittent systems (at least at low frequencies) show no isolated peaks. Intensity extends across a wide range of frequencies modulated by a power-law envelope.

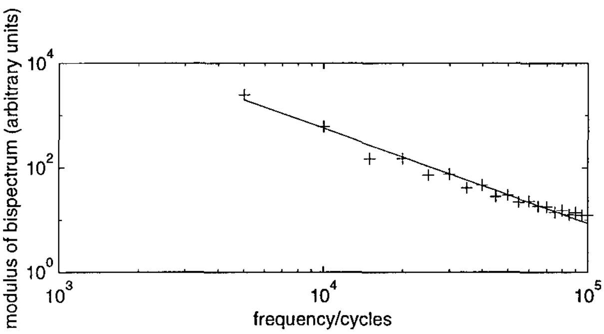

Figure 4. The theoretical treatment given in this paper predicts that the function $P(\omega_1, 0)$ should diverge as $\omega_1^{-2(1-\alpha)}$ as $\omega_1 \to 0$. This graph shows the theory (solid line) and values of a numerically calculated bispectrum using 10000 data points of a time series generated by the dynamical system of equation (28) for $z = 2.1$ ($\alpha = 1.909$).

These results suggest that bispectral measurements may be useful in cases where it is difficult to distinguish power-law noise from a dynamical system[16,2] from noise from a truly random source, such as filtered white noise. White noise retains its random phases when passed through a linear filter, and so has zero bispectrum.

5 Bispectra of financial time series

The bispectra of economic time series were first computed by Godfrey,[18] who was able to reject the null hypothesis of linearity for a number of stock price time series, but did not discuss other aspects of the bispectral form. For comparison with the other examples presented here, we calculated the bispectrum of the log-increments $y_n = \log(x_n/x_{n-1})$ of the Dow Jones Industrial Average stock market index, evaluated daily over the period January 1994 to February 1999. The bispectrum was estimated from the bicorrelation using a Parzen window, as in the work of Subba Rao and Gabr[7]. The result (figure 5) shows broad spectral intensity, as for the two chaotic examples presented here, but concentrated more at higher frequencies. This shift towards higher frequencies is probably a result of taking the log-increments, which is effectively a differentiation process.

6 Conclusion

One recognised characteristic of chaotic dynamical systems is that they show a continuous power spectrum, with intensity distributed across a wide range of frequencies. The results of this paper show that, at least in some cases, this is also true of their bispectra. This provides a way of distinguishing a chaotic time series from filtered white noise. However, other classes of processes share this characteristic of broad-band bispectral intensity. This is clear because the bispectra for the intermittent systems in this paper were calculated by exploiting the similarity

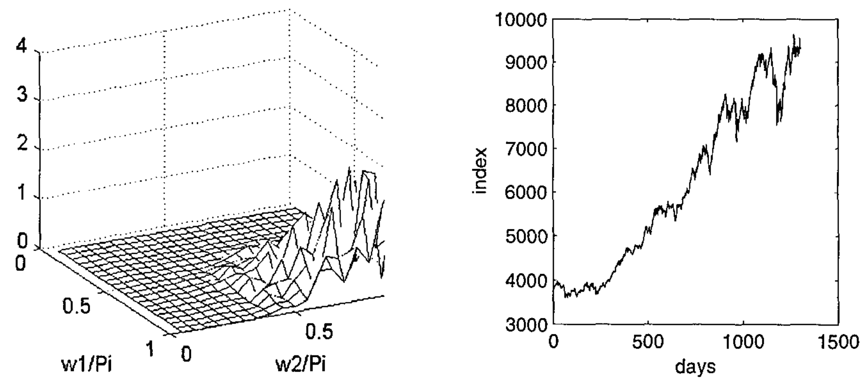

Figure 5. The left-hand graph shows the modulus of the bispectrum for the log-increments of the Dow Jones industrial average stock index (values shown on the right).

between these systems and renewal processes. A broad-band bispectrum can not therefore be taken as an unambiguous sign of chaos. A broad-band bispectrum does indicate interactions between Fourier modes over a wide range of frequencies, and is therefore a sign that the system studied does not have a simple description in the Fourier domain. The broad-band character of the bispectra of financial time series should therefore be taken as an indication of the complexity of the underlying processes.

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